## Fractals

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## Introduction to fractals-

A fractal is a pattern that never ceases to grow. Fractals are infinitely complicated patterns which are in almost all respects, self-similar. They are created by repeating a well defined set of rules over and over again i.e. driven by reiteration. They are images of dynamic systems - the results of chaos theory. Fractal patterns are not distant from us as the nature around us is filled with fractals.

- Sierpinski fractal is made up of three smaller triangles which are all exactly the same shape as the larger triangle. The prime idea is that they are in turn having exactly three similar triangles enclosed within. If seven sizes of triangles are cut out as in the image, it makes $3^{7}=2187$ small triangles already in the fractal.

- If a series of transformations scale any given point such as those on a circle's circumference or the mirror images of each pattern, then the fractal will have rotational or mirror symmetry, like the one below.
- These have been created using Fractal Geometry software using appropriate equations.

- Mandelbrot Set of fractals of the form $\mathbf{z n}_{\mathrm{n}+1}=\mathbf{z n}^{2} \mathbf{+} \mathbf{c}$



## Pascal's Triangle-

This triangular pattern is called Pascal's triangle after the French mathematician Blaise Pascal, who made many a breakthroughs in not only mathematics but physics and theology as well.

## Unique properties-

- Perfect squares of the natural numbers in any column are obtained by adding the number to its right and the one diagonally situated in the same direction.
Ex: $5^{2} \rightarrow 15+10$
$13^{2} \rightarrow 78+91=169$
- If we sum each row, we obtain powers of base 2 , beginning with $2^{0}=1$
- On rows that contain prime numbers alone, the numbers apart from the 1 s are divisible by the particular row number.
- Triangle also reveals powers of base 11.
- Combinatorial form which can be directly applied to Permutation and combination problems.

- Values in a row of Pascal's triangle can be taken as the coefficients of the binomial expansion $(x+y)^{n}$ and applied directly.

$$
\begin{array}{cc}
(x+y)^{0} & 1 \\
(x+y)^{1} & 1 x+1 y \\
(x+y)^{2} & 1 x^{2}+2 x y+1 y^{2} \\
(x+y)^{2} & 1 x^{3}+3 x^{2} y+3 x y^{2}+1 y^{3} \\
(x+y)^{4} & 1 x^{4}+4 x^{3} y+6 x^{2} y^{2}+4 x^{3}+1 y^{4} \\
(x+y)^{4} & 1 x^{5}+5 x^{4} y+10 x^{3} y^{2}+10 x^{2} y^{3}+5 y^{4}+1 y^{5}
\end{array}
$$

- Triangular numbers:

- Fibonacci sequence:

- Square of number:

Consider 5 in the $6^{\text {th }}$ row. Its square is equal to the sum of diagonally situated numbers 15 and 10.

- Cube of a number:


The cube of a number for example,
Consider 3, then taking one triangle of the star given, $4 \times 3=12$
Taking the other triangle, $2 \times 6=12$
Adding these to the number 3 gives $12+12+3=27$.

Sierpinski Triangle from Pascal's triangle:


Properties:

- A self-similar fractal that results from removing the triangle connecting the three midpoints of an equilateral triangle's sides, and continuing this process for the resulting equilateral triangles within that triangle.
- Sierpinski Gasket can be derived from it, which is collection of little squares drawn in a particular pattern within a square region.

- For pattern with number of dimensions $n$, when doubling a side of an object, $2^{n}$ copies of it are obtained, i.e. 2 copies for 1-dimensional object, 4 copies for 2 -dimensional object and 8 copies for 3-dimensional object and so on. In the Sierpinski triangle, doubling its side creates 3 copies of itself; this is presence of Hausdorff Dimensions.
- $\log (3) / \log (2)=\log _{2} 3 \approx 1.585$, which follows from solving $2^{n}=3$ for $n$ also. This proves the non integer fractal dimensional property of a fractal.

Finding the squares of two digit and three-digit numbers using a similar method:

- Take a two digit number, for example 76

Then, $76^{2}$ can be separated as

| $7^{2}$ | $2 \times 7 \times 6$ | $6^{2}$ |
| :---: | :---: | :---: |
| 49 | 84 | 36 |
| 4 | $9+8,4+3$ | 6 |
| 5 | 77 | 6 |

This now gives the result 5776 .
Another example-
Consider the number 49, on following the same method

| $4^{2}$ | $2 \times 4 \times 9$ | $9^{2}$ |
| :---: | :---: | :---: |
| 16 | 72 | 81 |
| 1 | $7+6,8+2$ | 1 |
| 2 | 40 | 1 |

This gives the result 2401.

- Let us now consider a three digit number say 126


## Rules:

\% We have to consider the unit digit and the digit at the ten's place as one number and proceed.

* Similarly, next two digits are to be considered/ taken together further.
* Only in between these pairs the 2 ab term is to be put.

So, $26^{2}$ has to be found separately. At times, the first method given below also works.

| $1^{2}$ | $2 \times 1 \times 2$ | $2^{2}$ | $6^{2}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 1 |  | 4 | 4 | 36 |
| 1 | 5 | 8 | 7 | 6 |


| $1^{2}$ | $2 \times 1 \times 26$ | $26^{2}$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 |  | 52 |  | 676 |  |  |
| 1 | 5 |  | $6+2$ | 7 |  |  |

This gives $126^{2}=15876$

- Another example, taking the number 389

| $3^{2}$ | $2 \times 3 \times 89$ | $89^{2}$ |  |  |
| :---: | :---: | :---: | :---: | :---: |
| 9 | 534 |  | 7921 |  |
| 9 | $534+79$ | 2 | 1 |  |
| $9+6$ |  | 13 | 2 | 1 |
| 15 | 13 | 2 | 1 |  |

This gives us $389^{2}=151321$

- Take the number 740,

| $7^{2}$ | $2 \times 7 \times 40$ | $40^{2}$ |  |
| :---: | :---: | :---: | :---: |
| 49 | 560 | 1600 |  |
| 49 | 5 | $60+16$ | 00 |
| 4 | $9+5$ | 76 | 00 |
| 5 | 4 | 76 | 00 |

So, $740^{2}=547600$

## Continued Fractions-

A continued fraction also represents a fractal $P / Q=[a ; b, c, d, \ldots]$ as it repeats infinitely within itself in different ways.

$$
\frac{P}{Q}=\frac{a+\frac{1}{b+\frac{1}{b+\frac{1}{d+\ldots}}}=a+1 /(b+1 /(c+1 /(d+\ldots)))}{}
$$

Continued fraction representation of quadratic irrationals- The simplest is for the golden ratio $\varphi$,

$$
\phi=\frac{1+\sqrt{5}}{2}=1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ddots}}}}
$$

Ramanujan came up with beautiful analogues to this using the base of the natural logarithms $e$.

$$
\frac{\sqrt{6 \sqrt{3}}-(1+\sqrt{3})}{4}=\frac{e^{-2 \pi / 3}}{1+\frac{e^{-2 \pi}+e^{-4 \pi}}{1+\frac{e^{-4 \pi}+e^{-8 \pi}}{1+\frac{e^{-6 \pi}+e^{-12 \pi}}{1+\ddots}}}}
$$

$$
\frac{\frac{1}{1+\frac{\mathrm{e}^{-2 \pi}}{1+\frac{\mathrm{e}^{-4 \pi}}{1+\frac{\mathrm{e}^{-6 \pi}}{1+\frac{\mathrm{e}^{-8 \pi}}{\mathrm{e}^{-10 \pi}}}}}}=\left(\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{1+\sqrt{5}}{2}\right) \mathrm{e}^{2 \pi / 5}}{1+\frac{\ddots}{\ddots}}{ }^{\frac{1}{2}}
$$

## Related puzzles-

1) Prove that the golden ratio

$$
\Phi=\frac{1+\sqrt{5}}{2}=1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ddots}}}}
$$

The continued fraction is self-similar all the way up to infinity. This means that wherever it is started to be solved, we end up with the same continued fraction. The continued fraction in the red box is exactly the same as that in the blue box. A part can be equal to the whole in numerals.

$$
\Phi=\frac{1+\sqrt{5}}{2}=1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ddots}}}}
$$

## Solution:

Based on this observed equality, an equation for the continued fraction is formed and solution is obtained using some elementary algebra. Setting the value of the blue box equal to $x$, we get $x=1+$ $1 / x$, which yields
$x^{2}-x-1=0$.

The solutions to this quadratic equation are:

$$
x=\frac{1 \pm \sqrt{5}}{2}
$$

2) Solve-


Let be equal to $x$,

$$
\frac{6}{1+\frac{6}{1+\frac{6}{1+. .}}}
$$

Then, $x=\frac{6}{1+x}$

$$
x^{2}+x-6=0
$$

On solving by the quadratic formula we get, $x=\frac{-1 \pm 5}{2}$

The continued fraction is equal to $\mathbf{2}$ or -3.

## Applications of fractals-

> They are seen all around us

$>$ Architecture
> Literature
$>$ Music
$>$ Nature and natural phenomena like water currents, galaxies, earthquakes etc.
> Poetry
> Psychology
> Recent technology - in Antennas

## References-

- www.quantamagazine.org
- https://science.howstuffworks.com/math-concepts
- www.study.com

